1. Let $x_0$ be a positive real number; we define a sequence of real numbers by setting:

$$x_{n+1} = x_n + \frac{16 - x_n^4}{4x_n^3}.$$  

Show that the sequence converges to 2.

2. Show that

$$\cos x \geq 1 - \frac{x^2}{2}$$

3. Evaluate the contour integral

$$I = \int_0^{2\pi} \frac{d\theta}{4 + \cos \theta}.$$  

4. What are the singularities of the function

$$f(z) = \frac{z e^{(1-z)^2}}{\sin \pi z}?$$

Label each as a pole, removable singularity, or essential singularity. Remember the “point at infinity.”

5. Let $(x(t), y(t))$ be a solution to the system of ordinary differential equations

$$\dot{x}(t) = -x(t) - y(t)$$
$$\dot{y}(t) = -y(t) + x(t).$$  

Prove that $\lim_{t \to \infty} (x(t), y(t)) = (0, 0)$.

6. For what values of $a$ is the following matrix positive definite

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & a \\ 1 & a & 2 \end{pmatrix}?$$

7. Suppose that we flip a fair coin $n$ times. Show that

(a) For any $0 < \epsilon$, show that the expected value of $(1 + \epsilon)^{\#\text{TAILS}}$ equals $(1 + \frac{\epsilon}{2})^n$.

(b) Use Chebyshev’s inequality to conclude that the probability of producing more than $\frac{2n}{3}$ TAILS tends to zero exponentially as $n \to \infty$. 
