1. Define a sequence of functions on \( \mathbb{R} \) by setting:

\[
f_n(x) = \frac{x}{1 + nx^2}
\]

Prove that \( < f_n(x) > \) converges uniformly on \( \mathbb{R} \) to a function \( f(x) \). For which \( x \) is it true that

\[
\lim_{n \to \infty} f_n'(x) = f'(x)?
\]

2. Suppose that \( f(z) \) is analytic in the disk of radius 2. What is the value of the contour integral

\[
\int_{|z|=1} f\left(\frac{1}{z}\right) \, dz?
\]

3. Consider the following game of chance: A circular target of radius 1 is divided into \( n \) concentric circles of radius \( 1/n, 2/n, \ldots, n/n = 1 \). A dart is tossed at random onto the circle; if it lands in the annular zone between the circles with radii \( k/n \) and \( (k+1)/n \), then \( n-k \) dollars are won, with \( k = 0, \ldots, n-1 \). Let \( X_n \) be a random variable denoting the amount of money won in one round of the game, and \( E(X_n) \) its expected value. Compute the limit

\[
\lim_{n \to \infty} \frac{E(X_n)}{n}.
\]

4. Suppose that \( A \) is an \( n \times n \) skew-symmetric matrix (\( A^t = -A \)). Prove that if \( n \) is an odd number, then there is a non-zero vector \( v_0 \) such that \( Av_0 = 0 \). Show that \( e^A \) is an orthogonal matrix, that is

\[
\langle e^A x, e^A y \rangle = \langle x, y \rangle,
\]

for any pair of vectors \( x, y \in \mathbb{R}^n \). What is \( e^A v_0 \)?

5. Suppose \( A \) is a symmetric, positive definite \( n \times n \) matrix. Compute the integral

\[
\int_{\mathbb{R}^n} e^{-\frac{1}{2}(Ax,x)} \, dx.
\]

Note that

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
\]

Hint: First diagonalize \( A \).
6. A Markov process with 2 states \{A, B\} is defined by the following transition probabilities:

\[
\begin{align*}
\text{Prob}(A|A) &= \frac{1}{3} & \text{Prob}(B|A) &= \frac{2}{3} \\
\text{Prob}(A|B) &= \frac{1}{2} & \text{Prob}(B|B) &= \frac{1}{2}
\end{align*}
\]

After many, many transitions what is the probability that the system will be found in state \(B\)?